The Little-Parks effect in an inhomogeneous superconducting ring

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Abstract

An inhomogeneous superconducting ring (hollow cylinder) placed in a magnetic field is considered. It is shown that the superconducting transition of the section with the lowest critical temperature may be a first order phase transition if the magnetic flux contained within the ring is not divisible by the flux quantum. In the vicinity of this transition, thermal fluctuations can induce the voltage in the ring with rather small sizes.

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1 Introduction

Superconductivity is a macroscopic quantum phenomenon. One of the consequences of this fact is the Little-Parks effect [1] which has been explained by M.Tinkham [2]. The experiment by Little and Parks is one of the first manifestations of the macroscopic quantum nature of superconductivity considered in many textbooks (see for example [3]). Little and Parks discovered that the critical temperature, T_c, of a superconducting tube with narrow wall depends in a periodic way on a magnetic flux value within the tube. This effect is a consequence of the dependence of the velocity of the superconducting electrons along the tube circumference on the magnetic flux value.

The velocity of the superconducting electrons v_s is given by (See ref.[3])

$$v_s = \frac{1}{m}(\hbar \frac{d\phi}{dr} - \frac{2e}{c}A) = \frac{2e}{mc}(\frac{\Phi_0}{2\pi}\frac{d\phi}{dr} - A) \tag{1}$$

where ϕ is the phase of the wave function $\Psi = |\Psi|\exp(i\phi)$ of the superconducting electrons; $\Phi_0 = \pi\hbar c/e$ is the flux quantum; A is the vector potential; m is the electron mass and e is the electron charge. As a consequence of this relation the velocity along the tube (or ring) circumference must have fixed values dependent on the magnetic flux because

$$\int_l dl v_s = \frac{2e}{mc}(\Phi_0 n - \Phi) \tag{2}$$

and $n = \int_l dl(1/2\pi)d\phi/dr$ must be an integer number since the wave function must be a simple function. Here $l = 2\pi R$ is the tube (or ring) circumference;

R is the tube (ring) radius; $\Phi = \int_l dlA$ is the magnetic flux contained within the ring. If the tube (ring) wall, w, is narrow enough (i.e. if $w \ll \lambda$), one has $\Phi \simeq BS$, because in this case the magnetic field induced by the superconducting current in the tube is small. Here B is the magnetic induction induced by an external magnet, $S = \pi R^2$ is the area of the tube cross-section and λ is the penetration depth of the magnetic field.

The energy of the superconductor increases with the superconducting electron velocity. Therefore the $|v_s|$ tends towards a minimum possible value. If Φ/Φ_0 is an integer number, this value is equal to zero. But v_s cannot be equal to zero if Φ/Φ_0 is not an integer number. Consequently, the energy of the superconducting state of the tube depends in a periodic manner on the magnetic field value. It is the cause of the Little-Parks effect.

If the tube (the ring) is homogeneous, then $\int_l dlv_s = 2\pi R v_s$. This case was considered in reference [2]. According to ref.[2], the critical temperature of the ring is shifted periodically in the magnetic field:

$$T_c(\Phi) = T_c[1 - (\xi(0)/R)^2(n - \Phi/\Phi_0)^2] \tag{3}$$

because the v_s^2 value changes periodically with magnetic field. Here $\xi(0)$ is the coherence length at T = 0. The value of $(n - \Phi/\Phi_0)$ changes from -0.5 to 0.5. The T_c shift is visible if the tube radius is small enough (if R is little more than $\xi(0)$.

In the present work an inhomogeneous ring (tube) with the narrow wall (the wall thickness $w \ll R, \lambda$) is considered. (In a ring the height h is smaller than the radius R (i.e. $h < R$) and in the tube $h > R$). Below the denomination "ring" is used in both cases. We consider a ring whose critical temperature varies along the circumference $l = 2\pi R$, but is constant along the height h. In such a ring, the magnetic flux shifts the critical temperature of a section with a lowest T_c value only. When the superconducting state is closed in the ring, the current of the superconducting electrons must appear as a consequence of the relation (2) if Φ/Φ_0 is not an integer number. Therefore the lowest T_c value will be shifted periodically in the magnetic field as well as T_c of the homogeneous ring. But in addition, the superconducting transition of the section with the lowest T_c may be a first order phase transition under some conditions. This principal feature of the inhomogeneous ring is considered in the present work.

2 Mean field approximation

Let us consider a ring consisting of two sections l_a and l_b ($l_a + l_b = l = 2\pi R$) with different values of the critical temperature $T_{ca} > T_{cb}$. According to the relation (2) the superconducting current along the ring circumference, I_s, must appear below T_{cb} if Φ/Φ_0 is not an integer number. It is obvious that the current value must be equal in both sections. Therefore if the current of the normal electrons is absent, the value of the superconductor currents I_{sa} and I_{sb} must be equal to

$$I_s = I_{sa} = s_a j_{sa} = s_a 2e n_{sa} v_{sa} = I_{sb} = s_b j_{sb} = s_b 2e n_{sb} v_{sb} \quad (4)$$

where n_{sa} and n_{sb} are the densities of the superconducting pair in the sections l_a and l_b; v_{sa} and v_{sb} are the velocities of the superconducting pairs in the sections l_a and l_b and s_a and s_b are the areas of wall section of l_a and l_b. We consider the ring with identical areas $s = s_a = s_b = wh$. $\int_l dl v_s = v_{sa} l_a + v_{sb} l_b$. Therefore according to (2) and (4)

$$v_{sa} = \frac{2e}{mc} \frac{n_{sb}}{(l_a n_{sb} + l_b n_{sa})} (\Phi_o n - \Phi); \; v_{sb} = \frac{2e}{mc} \frac{n_{sa}}{(l_a n_{sb} + l_b n_{sa})} (\Phi_o n - \Phi) \quad (5)$$

Consequently the v_{sb} value decreases with the n_{sb} value increasing. Therefore the dependence of the energy of the superconducting state on the n_{sb} value can have a maximum in some temperature region at $T \simeq T_{cb}(\Phi)$. The presence of such a maximum means that the superconducting transition is a first order phase transition.

The existence of the maximum and the width of the temperature region where the maximum exists depends on the $n - \Phi/\Phi_o$ value and on the ring parameters: l_a, l_b, w, h and T_{ca}/T_{cb}. We consider this dependence in the present paper. It is shown that these dependencies can be reduced to two parameters, B_f and L_I, which are introduced below. It is obvious that the maximum can exist at only $n - \Phi/\Phi_0 \neq 0$. Therefore only this case is considered below.

The hysteresis of the superconducting transition can be observed if the maximum is high enough. The maximum height is determined by a parameter F, which is introduced below. The hysteresis will be observed if the maximum height is much greater than the energy of the thermal fluctuation, $k_B T$. In the opposite case the thermal fluctuation switches the l_b section from the normal state into the superconducting one and backwards at $T \simeq T_{cb}(\Phi)$. This case is also considered below. It is shown that the switching between the normal and the superconducting states induces a voltage on the l_b section.

The Ginsburg-Landau free energy of the ring can be written as

$$F_{GL} = s[l_a((\alpha_a + \frac{m v_{sa}^2}{2}) n_{sa} + \frac{\beta_a}{2} n_{sa}^2) + l_b((\alpha_b + \frac{m v_{sb}^2}{2}) n_{sb} + \frac{\beta_b}{2} n_{sb}^2)] + \frac{L I_s^2}{2} \quad (6)$$

Here L is the inductance of the ring. $\alpha_a = \alpha_{a0}(T/T_{ca} - 1)$, β_a, $\alpha_b = \alpha_{b0}(T/T_{cb} - 1)$ and β_b are the coefficients of the Ginsburg-Landau theory. We do not consider the energy connected with the density gradient of the superconducting pair. It can be shown that this does not influence essentially the results obtained below.

The Ginsburg-Landau free energy (6) consists of $F_{GL,la}$ (the energy of the section l_a), $F_{GL,lb}$ (the energy of the section l_a) and F_L (the energy of the magnetic field induced by the superconducting current):

$$F_{GL} = F_{GL,la} + F_{GL,lb} + F_L \tag{7}$$

Substituting the relation (4) for the superconducting current and the relation (5) for the velocity of the superconducting electrons into the relation (6), we obtain

$$F_{GL,la} = sl_a(\alpha_a(\Phi, n_{sa}, n_{sb})n_{sa} + \frac{\beta_a}{2}n_{sa}^2) \tag{7a}$$

$$F_{GL,lb} = sl_b(\alpha_b(\Phi, n_{sa}, n_{sb})n_{sb} + \frac{\beta_b}{2}n_{sb}^2) \tag{7b}$$

$$F_L = \frac{2Ls^2e^2}{mc}\frac{(\Phi_0 n - \Phi)^2 n_{sa}^2 n_{sb}^2}{(l_a n_{sb} + l_b n_{sa})^2} \tag{7c}$$

Here

$$\alpha_a(\Phi, n_{sa}, n_{sb}) = \alpha_{a0}(\frac{T}{T_{ca}} - 1 + (2\pi\xi_a(0))^2\frac{(n - \Phi/\Phi_0)^2 n_{sb}^2}{(l_a n_{sb} + l_b n_{sa})^2})$$

$$\alpha_b(\Phi, n_{sa}, n_{sb}) = \alpha_{b0}(\frac{T}{T_{cb}} - 1 + (2\pi\xi_b(0))^2\frac{(n - \Phi/\Phi_0)^2 n_{sa}^2}{(l_a n_{sb} + l_b n_{sa})^2})$$

$\xi_a(0) = (\hbar^2/2m\alpha_{a0})^{1/2}$; $\xi_b(0) = (\hbar^2/2m\alpha_{b0})^{1/2}$ are the coherence lengths at T=0.

According to the mean field approximation the transition into the superconducting state of the section l_b occurs at $\alpha_b(\Phi, n_{sa}, n_{sb}) = 0$. Because $n_{sa} \neq 0$ at $T = T_{cb}$ the position of the superconducting transition of the l_b section depends on the magnetic flux value:

$$T_{cb}(\Phi) = T_{cb}[1 - (2\pi\xi_b(0))^2\frac{(n - \Phi/\Phi_0)^2 n_{sa}^2}{(l_a n_{sb} + l_b n_{sa})^2}] \tag{7d}$$

.

At $l_a = 0$ the relation (7d) coincides with the relation (3) for a homogeneous ring. A similar result ought be expected at $l_b \gg l_a$. But at $l_b \ll l_a$ the $T_{cb}(\Phi)$ value depends strongly on the n_{sb} value. At $n_{sb} = 0$ $T_{cb}(\Phi) = T_{cb}[1 - (2\pi\xi_b(0)/l_b)^2(n - \Phi/\Phi_0)^2]$ whereas at $l_a n_{sb} \gg l_b n_{sa}$ $T_{cb}(\Phi) = T_{cb}[1 - (2\pi\xi_b(0)/l_a)^2(n - \Phi/\Phi_0)^2]$. Consequently a hysteresis of the superconducting transition ought be expected in a ring for $l_b \ll l_a$.

To estimate the dependence of the hysteresis value on the ring parameters, we transform the relation (7) using the relations for the thermodynamic critical field $H_c = \Phi_0/2^{3/2}\pi\lambda_L\xi$; $\alpha^2/2\beta = H_c^2/8\pi$ and for the London penetration depth $\lambda_L = (cm/4e^2n_s)^{1/2}$. We consider a ring with $l_a \gg \xi_a(T) = \xi_a(0)(1-T/T_{ca})^{0.5}$. $n_{sa} \simeq -\alpha_a/\beta_a$ in this case. Then

$$F_{GL} = F_{GLa} + Fn'_{sb}(\tau + \frac{1}{(n'_{sb}+1)^2} + n'_{sb}(B + \frac{1}{(n'_{sb}+1)^2}(2 + L_I))) \tag{8}$$

Here $n'_{sb} = l_a n_{sb}/l_b n_{sa}$;

$$F_{GLa} = -sl_a \frac{H_{ca}^2}{8\pi}(1 + \frac{(2\pi\xi_a(T))^4}{l_a^4}(\frac{(n-\Phi/\Phi_0)n'_{sb}}{(n'_{sb}+1)})^4)$$

Because $l_a \gg 2\pi\xi_a$, $F_{GLa} \simeq -sl_a H_{ca}^2/8\pi$.

$$F = \frac{s\xi_a(T)H_{ca}^2}{2}\frac{2\pi\xi_a(T)}{l_a}(n-\Phi/\Phi_0)^2$$

$$\tau = (\frac{T}{T_{cb}} - 1)(n-\Phi/\Phi_0)^{-2}\frac{l_b^2}{(2\pi\xi_b(0))^2}$$

$$B_f = 0.5\frac{\beta_b}{\beta_a}\frac{l_b}{l_a}\frac{l_b^2}{(2\pi\xi_b(0))^2}(n-\Phi/\Phi_0)^{-2}$$

$$L_I = 4\pi\frac{s}{\lambda_{La}^2}\frac{L}{l_a}$$

For $h > R$, $L = k4\pi R^2/h$ where k = 1 at $h \gg R$. Consequently, $L_I = 4\pi(l/l_a)(lw/\lambda_{La}^2(T))$ in this case. At $h, w \ll R$, $L \simeq 4l\ln(2R/w)$, therefore $L_I = 16\pi(l/l_a)(s/\lambda_{La}^2(T))\ln(2R/w)$ in this case.

The numerical calculations show that the $F_{GL}(n'_{sb})$ dependence (8) has a maximum at small enough values of B_f and Ł$_I$ in some region of the τ values. The width of the τ region with the $F_{GL}(n'_{sb})$ maximum depends on the B_f value first of all. At Ł$_I \ll 2$ the maximum exists at $B_f < 0.4$. For example at $B_f = 0.2$ and $L_I \ll 2$ the maximum takes place at $-1.02 < \tau < -0.89$. This means that the transition into the superconducting state of the section l_b occurs at $\tau \simeq -1.02$, (that is at $T_{cs} = T_{cb}(1 - 1.02(n-\Phi/\Phi_0)^2(2\pi\xi_b(0)/l_b)^2)$ and the transition in the normal state occurs at $\tau \simeq -0.89$, (that is at $T_{cn} = T_{cb}(1 - 0.89(n-\Phi/\Phi_0)^2(2\pi\xi_b(0)/l_b)^2)$ if thermal fluctuations are not taken into account.

The inequality Ł$_I \ll 2$ is valid for a tube (when $h > R$) with $2\pi lw \ll \lambda_{La}^2(T)$ and for a ring (when $h < R$) with $8\pi hw \ll \lambda_{La}^2(T)$. The hysteresis value increases with decreasing B_f value and decreases with increasing B_f value. The B_f value is proportional to $(n-\Phi/\Phi_0)^{-2}$. Consequently, the hysteresis value depends on the magnetic field value. Because the hysteresis is absent at $B_f > 0.4$, it can be observed in the regions of the magnetic field values, where Φ/Φ_0 differs essentially from an integer number. The width of these regions depends on the $0.5(\beta_b/\beta_a)(l_b^3/(2\pi\xi_b(0))^2 l_a)$ value (see above the relation for B_f). Since $(n-\Phi/\Phi_0)^2 < 0.25$ and $\beta_b \simeq \beta_a$ in the real case, the hysteresis can be observed in the ring with $l_b^3 < 0.2(2\pi\xi_b(0))^2 l_a)$. For example in the ring with $l_b = 2\pi\xi_b(0)$ and $l_a = 10l_b$, the hysteresis can be observed at $|n-\Phi/\Phi_0| > 0.35$ (if $\beta_a = \beta_b$). At $|n-\Phi/\Phi_0| = 0.5$ $B_f = 0.2$ and the hysteresis is equal to $T_{cn} - T_{cs} \simeq 0.03T_{cb}$ in this ring.

3 Thermal fluctuation theory

Above we have used the mean field approximation which is valid when the thermal fluctuation is small. In our case the mean field approximation is valid if the height of the $F_{GL} - F_{GLa}$ maximum, $F_{GL,max}$, is much greater than $k_B T$. This height depends on the F, τ, B_f and L_I values: $F_{GL,max} = FH(\tau, B_f, L_I)$. The F parameter is determined above. The $H(\tau, B_f, L_I)$ dependence can be calculated numerically from the relation (8). To estimate the validity of the mean field approximation we ought to know the maximum value of the $H(\tau)$ dependence: $H_{max}(B_f, L_I)$. We can use the mean field approximation if $FH_{max}(B_f, L_I) \gg k_B T$. This is possible if the height of the ring is large enough, namely

$$h \gg \xi_a(0)\frac{1}{\pi H_{max}(B_f, L_I)}\frac{l_a}{w}\frac{Gi^{1/2}}{T_{ca}/T_{cb}-1}(n-\frac{\Phi}{\Phi_0})^{-2}$$

Here $Gi = (k_B T_{ca}/\xi_a(0)^3 H_{ca}^2)^2$ is the Ginsburg number of a three-dimensional superconductor. We have used the relation for the F parameter (see above). For conventional superconductors $Gi = 10^{-11} - 10^{-5}$. $H_{max} \simeq 10^{-2}$ for typical B_f and L_I values. For example in the ring with $B_f = 0.2$ and $L_I \ll 2$ the $H(\tau)$ dependence has a maximum $H_{max}(B_f = 0.2, L_I \ll 2) = 0.024$ at $\tau \simeq -0.94$. Consequently, the value of h cannot be very large. As an example for a ring with parameter value $B_f = 0.2$, $L_I \ll 2$, $l_a/w = 20$, $T_{ca}/T_{cb} - 1 = 0.2$, and fabricated from an extremely dirty superconductor with $Gi = 10^{-5}$, the mean field approximation is valid at $h \gg 20\xi_a(0)$ if $|n - \Phi/\Phi_0| \simeq 0.5$.

If the mean field approximation is not valid, we must take into account the thermal fluctuations which decrease the value of the hysteresis. The probability of the transition from normal into superconducting state and that of the transition from superconducting into normal state are large when the maximum value of $F_{GL} - F_{GLa}$ is no much more than $k_B T$. Therefore the hysteresis can not be observed at $FH_{max}(B_f, L_I) < k_B T$. This inequality can be valid for a ring made by lithography and etching methods from a thin superconducting film, where h is the film thickness in such a ring.

Let the l_b section of this ring have the lowest critical temperature T_{cb}. As a consequence of the thermal fluctuations, the density $n_{sb}(r,t)$ changes with time. We can consider $n_{sb}(r,t)$ as a function of the time only if $h, w, l_b \simeq \ or \ < \xi_b(T)$. At $T \simeq T_{cb}(\Phi)$ (at the resistive transition) l_b is switched by the fluctuations from the normal state in the superconducting state and backwards i.e. some times ($\simeq t_n$) $n_{sb} = 0$ and some times ($\simeq t_s$) $n_{sb} \neq 0$. According to (5) v_{sb} can not be equal to zero if $n_{sa} \neq 0$ and $\Phi_o n - \Phi \neq 0$. Therefore the superconducting current value I_s (see (4)) changes also with time.

The change of the superconducting current induces the change of the magnetic flux $\Phi = H\pi R^2 + L(I_s + I_n)$ and, as a consequence, induces the voltage and the current of the normal electrons (the normal current, I_n). The total current $I = I_s + I_n$ must be equal in both sections, because the capacitance is small.

But I_{sa} cannot be equal to I_{sb}. Then $I_{na} \neq I_{nb}$ in this case and consequently, the potential difference dU/dl exists along the ring circumference. Then the electric field along the ring circumference $E(r)$ is equal to

$$E(r) = -\frac{dU}{dl} - \frac{1}{l}\frac{d\Phi}{dt} = -\frac{dU}{dl} - \frac{L}{l}\frac{dI}{dt} = \frac{\rho_n}{s} I_n \tag{9}$$

where ρ_n is the normal resistivity. If the time of the normal state is much smaller than the decay time of the normal current, $t_n \ll L/R_{nb}$, the total current I is approximately constant in time. $R_{bn} = \rho_{bn} l_b/s$ is the resistance of the section l_b in the normal state. If also $t_s \gg L/R_{bn}$, then

$$I \simeq s2en_{sa} < v_{sa} > \simeq s\frac{4e^2}{mc}\frac{n_{sa} < n_{sb} >}{l_b n_{sa} + l_a < n_{sb} >}(\Phi_0 n - \Phi) \tag{10}$$

Here $< n_{sb} >$ is the thermodynamic average of n_{sb}. If $t_n \neq 0$, the resistivity of the l_b section, $\rho_b \simeq \rho_{bn} t_n/(t_s + t_n))$, is not equal to zero. Consequently the direct potential difference U_b can be observed on the section l_b at $T \simeq T_{cb}(\Phi)$

$$U_b = R_b I \simeq \frac{l_b < n_{sb} >}{l_b n_{sa} + l_a < n_{sb} >}\frac{(\Phi_0 n - \Phi)}{\lambda_{La}^2}\rho_b \tag{11}$$

According to (11) $U_b \neq 0$ at $< n_{sb} > \neq 0$ and $\rho_b \neq 0$. Consequently, the direct potential difference can be observed in the region of the resistive transition of the section l_b (where $0 < \rho_b < \rho_{bn}$).

The relation (11) is valid at $t_n \ll L/R_{bn}$. At $t_n \gg L/R_{bn}$ the direct part of the potential difference is equal to

$$U_b \simeq \frac{s < n_{sb} >}{l_b n_{sa} + l_a < n_{sb} >}\frac{(\Phi_0 n - \Phi)}{\lambda_{La}^2} L f \tag{12}$$

where f is the frequency of the switching from the normal into the superconducting state.

The Ginsburg-Landau free energy F_{GL} changes in time with amplitude $k_B T$ as a consequence of the thermal fluctuation. According to the relations (10),(12) and (7c) $U_b I/f \simeq F_L$. Because F_L is a part of F_{GL} (see (7)) the U_b value can not exceed $(k_B T R_{bn} f)^{1/2}$. The maximum value of the switching frequency f is determined by the characteristic relaxation time of the superconducting fluctuation τ_{GL}: $f_{max} \simeq 1/\tau_{GL}$. In the linear approximation region [4]

$$\tau_{GL} = \frac{\hbar}{8k_B(T - T_c)} \tag{13}$$

The width of the resistive transition of the section l_b can be estimated by the value $T_{cb} Gi_b$. $Gi_b = (k_B T/H_c^2(0) l_b s)^{1/2}$ is the Ginsburg number of the section l_b. Consequently the U_b value can not be larger than

$$U_{b,max} = (\frac{8R_b Gi_b}{\hbar})^{1/2} k_B T_{cb} \tag{14}$$

The $U_{b,max}$ value is large enough to be measured experimentally. Even at $T_c = 1\ K$ and for real values $R_b = 10\Omega$ and $Gi_b = 0.05$, the maximum voltage is equal to $U_{b,max} \simeq 3\mu V$. In a ring made of a high-Tc superconductor, $U_{b,max}$ can exceed $100\mu V$. One ought to expect that the real U_b value will be appreciably smaller than $U_{b,max}$. This voltage can be determined by the periodical dependence on the magnetic field value (see the relations (11) and (12)).

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